

In Class 11/12/2021

Last time:

Greene's Thm: Suppose you know D is a region in the plane w/ boundary a piecewise-smooth SCC. If $P(x,y)$ and $Q(x,y)$ have Cts partials on some open region R , containing all of D :

$$\int_{\partial D} P dx + Q dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$



ex: Compute

$$\int_C y^4 dx + 2xy^2 dy \text{ for}$$

C the positively ellipse $x^2 + y^2 = 2$

Sol:

$$\int y^4 dx + 2xy^2 dy$$

$$= \iint_D \frac{\partial}{\partial x} [2xy^2] - \frac{\partial}{\partial y} [y^4] dA$$

$$= \iint_D 2y^2 - 4y^3 dA$$



need to parameterize D $\hookrightarrow 0 \leq \theta \leq 2\pi$

$$2x^2 + 2y^2 = 2$$

\uparrow would be circle so need to substitute

$$x = \sqrt{2} r \cos \theta$$

$$y = r \sin \theta$$

brings in that 2!! and makes polar

* plug into equation

$$2r^2 \cos^2 \theta + 2r^2 \sin^2 \theta = 2$$

$$\hookrightarrow 2r^2 = 2$$

$$0 \leq r \leq 1$$

UNIT CIRCLE

$$\int_0^{2\pi} \int_{r=0}^1 2y^2 - 4y^3$$

$$\sqrt{2} r \, dr \, d\theta$$

& Jacobian

inner:

$$2\sqrt{2}r^3 \int_0^{2\pi} \sin^2 \theta (1 - 2r \sin \theta) \, d\theta$$

$$= \int_0^{2\pi} (1 - \cos^2 \theta) (1 - 2r \sin \theta) \, d\theta$$

$$\frac{d(x,y)}{d(r,\theta)} = \det \begin{bmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} \cos \theta & -\sqrt{2} r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

$$r \sqrt{2} \cos^2 \theta + r \sqrt{2} \sin^2 \theta = \sqrt{2} r$$

$$\frac{1}{\sqrt{2}}$$

$$\int_0^{2\pi} (1 - \cos^2 \theta) \, d\theta - 2r \int_0^{2\pi} (1 - \cos^2 \theta) \sin \theta \, d\theta$$

$$\begin{aligned}
&= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2\theta) \right) d\theta + 2r \left[u - \frac{1}{3} u^3 \right]_0^{2\pi} \\
&= \left[\frac{1}{2} \theta + \frac{1}{4} \sin(2\theta) + 2r \left[\cos \theta - \frac{1}{3} \cos^3 \theta \right] \right]_0^{2\pi} \\
&= \frac{1}{2} (2\pi - 0) - \frac{1}{4} (\cancel{\sin 4\pi} - \cancel{\sin 0}) + 2r \left(\cancel{\cos 2\pi} - \frac{1}{3} \cancel{\cos^3 2\pi} - \cancel{\cos 0} + \frac{1}{3} \cancel{\cos^3 0} \right) \\
&= \pi
\end{aligned}$$

$$\begin{aligned}
&\int_{r=0}^1 2\sqrt{2} r^3 \pi \, dr \\
&= \frac{2\sqrt{2} \pi}{4} r^4 \Big|_0^1 = \boxed{\frac{\pi\sqrt{2}}{2}}
\end{aligned}$$

If P, Q satisfy $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,

then by Green's thm:

$$\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \text{Area}(D)$$

can compute w/
line integral!

ex: Compute A of gen. ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol: Area(D) = $\int_{\partial D} P dx + Q dy$, if we chose

$$P, Q \text{ w/ } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \quad \leftarrow$$

$$\text{chose } Q=0, P=-y \text{ i } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 + 1 = \boxed{1}$$

$$\text{Area}(D) = \int_0 -y dx + 0 dy = \int -y dx$$

need parameters

The ellipse ∂D is parameterized by
 $\vec{r}(t) = \langle a \cos(t), b \sin(t) \rangle$ on

$$0 \leq t \leq 2\pi \quad dx = x'(t) dt$$

$$- \int_{t_0}^{2\pi} b \sin(t) - a \sin(t)$$

$$= ab \int_{t=0}^{2\pi} \sin^2(t) dt$$

$$= ab \int_0^{2\pi} \frac{1}{2} (1 - \cos(2t)) dt$$

$$= \frac{1}{2} ab \left[t - \frac{1}{2} \sin(2t) \right]_{t=0}^{2\pi} = \frac{1}{2} ab (2\pi - 0) = \boxed{ab\pi}$$

§16.5 Curl and Divergence

Goal: define and study operations on vector fields

Defn: The curl of a vector field \vec{v} on \mathbb{R}^3 (exactly 3 components) is $v = \langle P, Q, R \rangle$

$$\text{Curl}(\vec{v}) = \nabla \times \vec{v} =$$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle$$

$$= \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

ex: compute $\text{curl}(\vec{v})$ for $\vec{v} = \langle xy, xyz, -y^2 \rangle$

sol $\text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xyz & -y^2 \end{vmatrix}$

* just a cross product

$$\begin{aligned}
 &= \left(\frac{d}{dy} [-y^2] - \frac{d}{dy} [xy^2], \frac{d}{dz} [xy] - \frac{d}{dx} [-y^2], \frac{d}{dx} [xy^2] - \frac{d}{dy} [xy] \right) \\
 &= \langle -2y - xy, 0 - 0, y^2 - x \rangle \\
 &= \boxed{\langle -xy - 2y, 0, y^2 - x \rangle}
 \end{aligned}$$

otherwise: Suppose we had a conservative v.f.
 $\vec{v} = \langle f_x, f_y, f_z \rangle$

$$\text{So } \text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f_x & f_y & f_z \end{bmatrix}$$

$$= \left\langle \frac{df_z}{dy} - \frac{df_y}{dz}, \frac{df_x}{dz} - \frac{df_z}{dx}, \frac{df_y}{dx} - \frac{df_x}{dy} \right\rangle$$

via Clairaut's thm:

$$\langle \underbrace{f_{zy}}_{=} - \underbrace{f_{yz}}_{=}, \underbrace{f_{xz}}_{=} - \underbrace{f_{zx}}_{=}, \underbrace{f_{yx}}_{=} - \underbrace{f_{xy}}_{=} \rangle$$

therefore $\boxed{\vec{0}}$

$$\boxed{\text{curl}(\nabla f) = \vec{0}}$$

* curl of conservative vector field is zero

* Proposition: A v.f. \vec{v} , comprising its particles is conservative iff $\text{curl}(\vec{v}) = \vec{0}$

Defn: The divergence of a v.f. $\vec{v} = \langle v_1, \dots, v_n \rangle$ is

$$\text{div}(\vec{v}) = \nabla \cdot \vec{v} = \left\langle \frac{d}{dx}, \dots, \frac{d}{dx_n} \right\rangle \cdot \langle v_1, v_2, \dots \rangle$$

$$= \sum_{i=1}^n \frac{dv_i}{dx_i}$$

ex: for $\vec{v} = \langle xy, xyz, -y^2 \rangle$

$$\text{div}(\vec{v}) = \frac{d}{dx} [xy] + \frac{d}{dy} [xyz] + \frac{d}{dz} [-y^2]$$

$$= y + xz + 0 = \boxed{y + xz}$$

Suppose: $\vec{v} = \text{curl}(\vec{w}), \vec{w} = \langle P, Q, R \rangle$

$$\vec{v} = \langle R_y - Q_z, -R_x - P_z, Q_x - P_y \rangle$$

$$\text{div}(\vec{v}) = \frac{d}{dx} [R_y - Q_z] + \frac{d}{dy} [-R_x - P_z] + \frac{d}{dz} [Q_x - P_y]$$

$$= R_{yx} - Q_{zx} - R_{xy} + P_{zy} + Q_{xz} - P_{yz}$$

$$= (P_{zy} - P_{yz}) + (Q_{xy} - Q_{yx}) + (R_{yz} - R_{zy})$$

via Clairauts

$$= 0 + 0 + 0 = \boxed{0} \quad * \# \text{ not vector}$$

* - A vf is the curl of another vf
 iff its divergence is zero
 → above, showed $\text{div}(\text{curl}(\vec{v})) = 0$